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THE PLANE STATE OF STRESS OF AN ELASTIC PLANE WITH TWO INTERSECTING SLITS*

YU.A. ANTIPOV and G.YA. POPOV

The construction of an exact closed solution for the problem of stress concentration in an elastic plane near two rectilinear slits of identical length that intersect at the centre at an arbitrary angle is proposed. An arbitrary rupturing and shearing load is applied along the slit edges. The construction of the solution of the problem is based on its reduction to a Riemann problem with a matrix coefficient of special structure that allows solution by quadratures. The possibility of solving such a problem was mentioned in /1/. This solution was first constructed for the case when the index $\varkappa_\varepsilon = 0$ for the ratio of the characteristic functions of the matrix coefficient /2/, and then also for $\varkappa_\varepsilon \neq 0$ /3/.

A different method from that described in /3/ is proposed for solving the Riemann problem for the cases when $\varkappa_\varepsilon = 1$ and $\varkappa_\varepsilon = -1$. The solution of the problem, constructed by quadratures, is converted to a form convenient for numerical realization. Computational formulas are obtained for the stress intensity factors.

1. Formulation of the problem of intersecting slits and its separation into an auxiliary problem. We investigate the plane state of stress of an elastic plane with two slits of identical length $2b$ (without loss of generality, we consider $b = 1$) that intersect at the centre at an arbitrary angle 2α (problem T). We take the bisectrix of the large angle between the slits as the horizontal axis of symmetry (Fig.1). As usual, the positive direction of variation of the angle θ is counter-clockwise. A positive load

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$\sigma_\theta = f_n(r)$, $\tau_{r\theta} = h_n(r)$ $(n = 1, 2, 3, 4)$ is applied to the slit edges OA_n.

We separate problem T into a symmetric and antisymmetric one relative to the horizontal axis of symmetry, and then each of the problems obtained into two more: symmetric and antisymmetric relative to the vertical axis of symmetry. Then the solution of problem T is the sum of the solution of four problems T_{kj} $(k = 1, 2;\ j = 1, 2)$, each of which is formulated as follows.

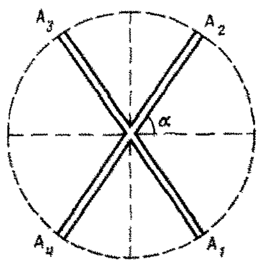

Fig.1

An infinite wedge $(0 < r < \infty,\ \alpha - \pi/2 < \theta < \alpha)$ has a slit $(0 < r < 1,\ \theta = \pm 0)$ (Fig.2) to whose edges a load

$$\sigma_\theta(r, \pm 0) = f_{kj}(r), \quad \tau_{r\theta}(r, \pm 0) = h_{kj}(r) \quad (0 < r < 1)$$

$$f_{kj} = [f_1 - (-1)^k f_2 + (-1)^{k+j} f_3 - (-1)^j f_4]/4$$

$$h_{kj} = [h_1 + (-1)^k h_2 + (-1)^{k+j} h_3 + (-1)^j h_4]/4$$

is applied. On the wedge face $\theta = \alpha$ the k-th boundary condition and on the face $\theta = \alpha - \pi/2$ the j-th boundary condition are satisfied from the following set $(0 < r < \infty)$:

$$1)\ u_\theta = \tau_{r\theta} = 0; \qquad 2)\ u_r = \sigma_\theta = 0$$

For instance, condition 1) should be satisfied on the fact $\theta = \alpha$ for problem T_{12} and condition 2) on the face $\theta = \alpha - \pi/2$.

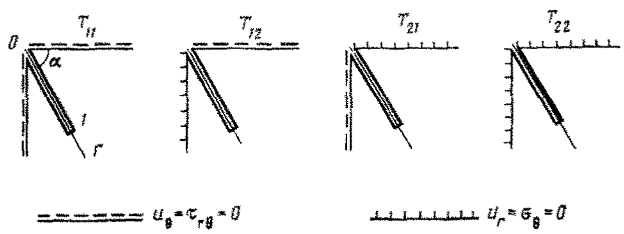

Fig.2

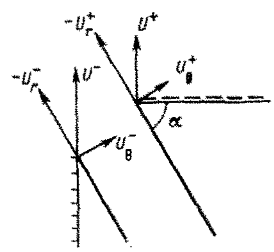

Fig.3

The following relationships hold for $0 < r < \infty$

$$\sigma_\theta|_{\theta=-0} = \sigma_\theta|_{\theta=+0}, \quad \tau_{r\theta}|_{\theta=-0} = \tau_{r\theta}|_{\theta=+0}$$
$$\left.\frac{\partial u_\theta}{\partial r}\right|_{\theta=-0} - \left.\frac{\partial u_\theta}{\partial r}\right|_{\theta=+0} = \chi_1(r), \quad \left.\frac{\partial u_r}{\partial r}\right|_{\theta=-0} - \left.\frac{\partial u_r}{\partial r}\right|_{\theta=+0} = \chi_2(r) \tag{1.1}$$

The functions χ_1 and χ_2 equal zero for $1 < r < \infty$ and are unknown for $0 \leqslant r \leqslant 1$. The additional condition

$$\int_0^1 \chi_1(r)\,dr = \operatorname{tg}\left(\alpha - \frac{k-1}{2}\pi\right) \int_0^1 \chi_2(r)\,dr \tag{1.2}$$

should be satisfied in problems T_{kj} $(k = 1,\ j = 2$ or $k = 2,\ j = 1)$.

We will prove this condition for $k = 1$ (problem T_{12}). Using the notation $u_\theta^\pm = u_\theta|_{\theta=\pm 0}$, $u_r^\pm = u_r|_{\theta=\pm 0}$ for $r = 0$, by virtue of (1.1) we have

$$\int_0^1 \chi_1(r)\,dr = u_\theta^+ - u_\theta^-, \quad \int_0^1 \chi_2(r)\,dr = u_r^+ - u_r^- \tag{1.3}$$

Furthermore, since the displacements $u^\pm$ equal zero in the direction $\theta = \alpha + \pi/2$ for $\theta = \pm 0$, $r = +0$ (Fig.3), we have

$$u^\pm \equiv u_\theta^\pm \cos\alpha - u_r^\pm \sin\alpha = 0$$

Comparing the last relationship with (1.3), we arrive at the required condition (1.2). It is proved analogously for $k = 2$.

Let $L^{(n)}$ and $N^{(n)}$ be stress intensity factors, respectively of σ_θ and $\tau_{r\theta}$ at the vertex A_n (Fig.1) for the problem T while L_{kj} and N_{kj} are the stress intensity factors of σ_θ and $\tau_{r\theta}$ for problem T_{kj}. Then it can be seen that

$$L^{(1)} = L_{11} + L_{12} + L_{21} + L_{22}, \quad N^{(1)} = N_{11} + N_{12} + N_{21} + N_{22}$$

$$L^{(2)} = L_{11} + L_{12} - L_{21} - L_{22},\ N^{(2)} = -N_{11} - N_{12} + N_{21} + N_{22}$$
$$L^{(3)} = L_{11} - L_{12} - L_{21} + L_{22},\ N^{(3)} = N_{11} - N_{12} - N_{21} + N_{22}$$
$$L^{(4)} = L_{11} - L_{12} + L_{21} - L_{22},\ N^{(4)} = -N_{11} + N_{12} - N_{21} + N_{22}$$

It is established from an analysis of the characteristic equations of problems T_{kj} that all the problems formulated from class S /4/ (it consists of problems for which the Saint-Venant principle is valid) and the stresses decrease as $r \to \infty$ no more slowly than r^{-1}. As $r \to 0$ the stresses can increase but not more rapidly than $r^{-\delta}$ $(0 < \delta < 1)$.

2. Solution of problems T_{kk}. We will first obtain the solution of the auxiliary problems T_{kk} $(k = 1, 2)$. We introduce the vectors

$$\chi(r) = \|\chi_1(r),\ \chi_2(r)\|,\ \psi(r) = \|\sigma_\theta(r, 0),\ \tau_{r\theta}(r, 0)\| \tag{2.1}$$
$$\chi_-^\circ(s) = \int_0^1 \chi(r) r^s\, dr,\quad \psi_-^\circ(s) = \int_0^1 \psi(r) r^s\, dr,\quad \psi_+^\circ(s) = \int_1^\infty \psi(r) r^s\, dr$$

The vectors $\chi_-^\circ(s)$ and $\psi_-^\circ(s)$ are analytic functions for $\operatorname{Re} s > -\delta$, and $\psi_+^\circ(s)$ for $\operatorname{Re} s < 0$. Let $L = L_\gamma^- \cup C_\gamma \cup L_\gamma^+$, $L_\gamma^\pm = \{t \in \mathbf{C}:\ \operatorname{Re} t = -0,\ \operatorname{Im} t \gtrless \pm|\gamma|\}$, $C_\gamma = \{t \in \mathbf{C},\ |t| = |\gamma|,\ \operatorname{Re} t < 0\}$, $-\delta < \gamma < 0$. The contour L divides the plane of the complex variable $\mathbf{C}$ into two domains D^+ and $D^- \ni 0$. We consider the positive direction on L to be that which keeps the domain D^+ to the left. Since $\psi(r) = O(r^{-1})$ as $r \to \infty$ and

$$\lim_{\gamma\to -0} \int_1^\infty \frac{1}{r} r^{\gamma + i\tau}\, dr = \lim_{\gamma \to -0}\left(\frac{-\gamma}{\gamma^2+\tau^2} + \frac{i\tau}{\gamma^2+\tau^2}\right) = \pi\delta(\tau) + \frac{i}{\tau}$$

holds ($\delta(\tau)$ is the delta function), then the vector $\psi_+^\circ(s)$ is analytic in the domain $D^+ \cup L$ while the vectors $\chi_-^\circ(s)$ and $\psi_-^\circ(s)$ are analytic in $D^- \cup L$. Therefore, the vector $\varphi(s) = \|\varphi_1(s),\ \varphi_2(s)\|$ defined by the relationships

$$\varphi(s) = \psi_+^\circ(s),\quad s \in D^+;\quad \varphi(s) = \chi_-^\circ(s),\quad s \in D^- \tag{2.2}$$

is piecewise-analytic with the line of jumps L.

We reduce problems T_{kk} to a Riemann matrix problem by the method /5/ based on the Mellin integral transform (E is the elastic modulus)

$$\varphi^-(t) = a \operatorname{ctg}(\pi t/2) G(t) \varphi^+(t) + g(t),\quad t \in L \tag{2.3}$$
$$a = 4/E,\quad g(t) = a \operatorname{ctg}(\pi t/2) G(t) \psi_-^\circ(t)$$

$$G(s) = \left\|\begin{matrix} b(s) + c(s)l(s) & c(s)m_-(s) \\ c(s)m_+(s) & b(s) - c(s)l(s)\end{matrix}\right\| \tag{2.4}$$

$$b(s) = {}^1/_2(-1)^{k+1} \operatorname{tg}(\pi s/2)[d_k^{-1}(s, \alpha)\cos 2\alpha s + d_k^{-1}(s, \pi/2 - \alpha)\cos(\pi - 2\alpha)s]$$
$$c(s) = {}^1/_2 \operatorname{tg}(\pi s/2)[d_k^{-1}(s, \alpha) - d_k^{-1}(s, \pi/2 - \alpha)]$$
$$d_k(s, \alpha) = s \sin 2\alpha - (-1)^k \sin 2\alpha s \quad (k = 1, 2) \tag{2.5}$$
$$l(s) = -\cos 2\alpha,\quad m_\pm(s) = (\pm s + 1)\sin 2\alpha$$

Let

$$\Delta(s) = \lambda_1(s)\lambda_2(s),\quad \varepsilon(s) = \frac{1}{2}\ln\frac{\lambda_1(s)}{\lambda_2(s)},\quad f(s) = l^2(s) + m_+(s)m_-(s)$$
$$\varkappa_\varepsilon = \operatorname{ind}\{\lambda_1(t)[\lambda_2(t)]^{-1}\},\quad \varkappa_\Delta = \operatorname{ind}[\lambda_1(t)\lambda_2(t)]$$

where $\lambda_1(s), \lambda_2(s)$ are characteristic functions (eigennumbers) of the matrix $G(s)$; then $\Delta(s)$ and $\varepsilon(s)$ are the determinant and index of $G(s)$ /2/ and

$$\lambda_n(s) = b(s) - (-1)^n c(s) f^{1/2}(s) \quad (n = 1, 2)$$

The characteristic functions possess the following directly confirmable properties

1) $\lambda_n(-0 + i\tau) \sim 1,\quad \tau \to \pm\infty$

2) $\lambda_n(\gamma) > 0,\quad \gamma \to -0$

3) $\lambda_n(t) > 0,\ t \in L_\gamma^\pm \quad (t = i\tau,\ |\gamma| < |\tau| < \infty)$

It follows from these properties that $[\arg \lambda_n(t)]|_L = 0$, meaning $\varkappa_\varepsilon = \varkappa_\Delta = 0$. Therefore, the solution of the factorization problem

$$G(T) = X^+(t)[X^-(t)]^{-1} = [X^-(t)]^{-1}X^+(t),\quad t \in L \tag{2.6}$$

is determined by the formulas /2/

$$\frac{X(s)}{\Lambda(s)}=\left\|\begin{matrix} c_+(s) & s_+(s) \\ s_-(s) & c_-(s)\end{matrix}\right\|, \quad \Lambda(s)=\exp\left[\frac{1}{2\pi i}\int_L \frac{\ln \Delta^{1/2}(t)}{t-s}\,dt\right] \tag{2.7}$$

$$c_\pm(s) = \operatorname{ch}[f^{1/2}(s)\beta(s)] \pm l(s)f^{-1/2}(s)\operatorname{sh}[f^{1/2}(s)\beta(s)] \tag{2.8}$$

$$s_\pm(s) = m_\mp(s)f^{-1/2}(s)\operatorname{sh}[f^{1/2}(s)\beta(s)]$$

$$\beta(s)=\frac{1}{2\pi i}\int_L \frac{\varepsilon(t)}{f^{1/2}(t)}\frac{dt}{t-s} \tag{2.9}$$

We use the representation

$$\operatorname{tg}\frac{\pi s}{2} = -\frac{K^+(s)}{K^-(s)}; \quad K^+(s)=\frac{\Gamma(1/2-s/2)}{\Gamma(-s/2)}, \quad K^-(s)=\frac{\Gamma(1+s/2)}{\Gamma(1/2+s/2)}$$

where the functions $K^\pm(s)$ are analytic in the domains $D^\pm$ and have no zeros there, where we have by virtue of the expansion from /6/ (p.62)

$$K^\pm(s) \sim (\mp s/2)^{1/2}, \quad s\to\infty, \quad s\in D^\pm \quad (|\arg s| < \pi)$$

On the basis of a theorem of Abelian type /7/ (p.473), we obtain $\varphi(s) = O(s^{-1/2})$, $s\to\infty$, $s\in D^\pm$ $(s \overline{\in} L)$. We substitute (2.6) into (2.3) and take into account the boundedness of the canonical matrix $X(s)$ at infinity. Subsequent, application of Liouville's theorem results in the following solution of problem (2.3):

$$\varphi^+(s)=-\frac{K^+(s)}{aX^+(s)}\Omega^+(s), \quad s\in D^+; \quad \varphi^-(s)=\frac{K^-(s)}{X^-(s)}\Omega^-(s), \quad s\in D^-$$

$$\Omega(s)=\frac{a}{2\pi i}\int_L \frac{X^+(t)}{K^+(t)}\frac{\psi_-^\circ(t)}{t-s}\,dt$$

We will now determine the stress intensity factors $\sigma_\theta(r,0)$, $\tau_{r\theta}(r,0)$ at the apex of the slit

$$K_1=\lim_{r\to 1+0}\sqrt{2\pi(r-1)}\,\sigma_\theta(r,0), \quad K_2=\lim_{r\to 1+0}\sqrt{2\pi(r-1)}\,\tau_{r\theta}(r,0) \tag{2.10}$$

Because of (2.1) and (2.2) we have on the basis of a theorem of Abelian type

$$\varphi_n^+(s) \sim K_n(-2s)^{-1/2}, \quad s\to\infty, \quad |\arg(-s)| \leqslant \theta_0 < \pi/2$$

Taking into account that

$$\varepsilon(i\tau)f^{-1/2}(i\tau) = O(e^{-p_0|\tau|}), \quad |\tau|\to\infty \quad (p_0>0)$$

we obtain an estimate for the integral (2.9)

$$\beta(s)\sim -\frac{u}{s\sin 2\alpha}, \quad s\to\infty; \quad u=\frac{\sin 2\alpha}{\pi}\int_0^\infty \frac{\varepsilon(i\tau)\,d\tau}{f^{1/2}(i\tau)}$$

and then taking account of (2.7) we have

$$[X(s)]^{-1}\sim Q, \quad s\to\infty; \quad Q=\left\|\begin{matrix}\cos u & -\sin u\\ \sin u & \cos u\end{matrix}\right\|$$

If we use the notation

$$v=\lim_{s\to\infty}\frac{s}{a}\Omega(s)=-\frac{1}{2\pi i}\int_L \frac{X^+(t)}{K^+(t)}\psi_-^\circ(t)\,dt$$

we finally find

$$K = Qv, \quad K = \|K_1, K_2\| \tag{2.11}$$

This formula determines the factors L_{kk}, N_{kk} for problems T_{kk} $(k=1,2)$: $L_{kk}=K_1$, $N_{kk}=K_2$.

Let us write (2.11) in detail for the constant load case, i.e., $\psi(r)\equiv\psi=\|\psi_1,\psi_2\|$. We have

$$\Omega(s)=\frac{a}{s+1}\left[\frac{X^+(t)\psi}{K^+(s)}-\frac{X^+(-1)\psi}{K^+(-1)}\right], \quad s\in D^+$$

$$K = -\sqrt{\pi} Q X^{+}(-1)\psi$$

Analogous formulas can be obtained for a polynomial load as well.

We note that it is convenient to use the following formulas for a numerical realization:

$$\beta(s) = -\frac{s}{\pi}\int_0^\infty \frac{\varepsilon(i\tau)}{f^{1/2}(i\tau)}\frac{dt}{\tau^2+s^2}, \quad \Lambda(s) = \exp\left[-\frac{s}{2\pi}\int_0^\infty \frac{\ln\Delta(i\tau)}{\tau^2+s^2}d\tau\right]$$

$(s \neq 0)$, which are obtained from (2.9) and (2.7) by obvious reduction /2/.

3. **Solution of problem T_{12}.** We now examine the case when $k \neq j$. Let $k = 1, j = 2$, first. Retaining the same notation of (2.1) and (2.2) as in the solution of problems T_{kk}, we arrive at the Riemann matrix problem

$$\varphi^-(t) = -aiG(t)\varphi^+(t) + g(t), \quad t \in L \tag{3.1}$$

where $a = 4/E$, $g(t) = -aiG(t)\psi_-^\circ(t)$, and $G(s)$ has the form (2.4). where

$$b(s) = {}^1/_2 i\,[d_1^{-1}(s, \alpha)\cos 2\alpha s - d_2^{-1}(s, \pi/2-\alpha)\cos(\pi - 2\alpha)s]$$

$$c(s) = {}^1/_2 i\,[d_1^{-1}(s, \alpha) - d_2^{-1}(s, \pi/2-\alpha)]$$

$$l(s),\ m_\pm(s)$$

and $d_k(s, \alpha)$ are defined in (2.5).

The characteristic functions possess the following properties

$$\begin{aligned} &1)\ \lambda_n(-0+i\tau) \sim \pm 1, \quad \tau \to \pm\infty \quad (n = 1, 2) \\ &2)\ \lambda_1(\gamma) \sim i\pi(\eta\gamma)^{-1}, \quad \lambda_2(\gamma) \sim -i\pi\gamma/4, \quad \gamma \mapsto -0 \\ &\eta = (2\alpha + \sin 2\alpha)(\pi - 2\alpha - \sin 2\alpha) > 0 \quad (0 < \alpha < \pi/2) \\ &3)\ \lambda_n(t) \lessgtr 0, \quad t \in L_\gamma^\mp \quad (n = 1, 2) \end{aligned} \tag{3.2}$$

Hence, $[\arg\lambda_1(t)]|_L = \pi$, $[\arg\lambda_2(t)]|_L = -\pi$, and consequently

$$\varkappa_\Delta = \operatorname{ind}\{\lambda_1(t)\lambda_2(t)\} = 0, \quad \varkappa_\varepsilon = \operatorname{ind}\{\lambda_1(t)[\lambda_2(t)]^{-1}\} = 1$$

We select the branch of the logarithms of the characteristic functions

$$0 \leqslant \operatorname{Im}\{\ln\lambda_n(t)\} \leqslant 2\pi, \quad t \in L$$

$$\operatorname{Im}\{\ln\lambda_1(t)\} = \begin{cases}\pi, & t \in L_\gamma^- \\ 2\pi, & t \in L_\gamma^+\end{cases}, \quad \operatorname{Im}\{\ln\lambda_2(t)\} = \begin{cases}\pi, & t \in L_\gamma^- \\ 0, & t \in L_\gamma^+\end{cases}$$

Then

$$\theta_\varepsilon(t) = \arg\frac{\lambda_1(t)}{\lambda_2(t)} = \begin{cases}0, & t \in L_\gamma^- \\ 2\pi, & t \in L_\gamma^+\end{cases} \tag{3.3}$$

$$0 \leqslant \theta_\varepsilon(t) \leqslant 2\pi, \quad t \in C_\gamma$$

Therefore, the branch of the index $\varepsilon(t)$ and the function $\ln\Delta^{1/2}(t)$ are found

$$\varepsilon(t) = \frac{1}{2}\ln\left|\frac{\lambda_1(t)}{\lambda_2(t)}\right| + \frac{i}{2}\theta_\varepsilon(t), \quad t \in L \tag{3.4}$$

$$\ln\Delta^{1/2}(t) = {}^1/_2\ln|\lambda_1(t)\lambda_2(t)| + i\pi, \quad t \in L \tag{3.5}$$

To make the subsequent intermediate calculation specific, we fix the branch of the function $f^{1/2}(s) = (b_1^2 - b_2^2 s^2)^{1/2}$, $b_1 = 1$, $b_2 = \sin 2\alpha$ (as for the solution of problems T_{kk} the selection of the branch does not influence the final formulas governing the solution of the problem). We construct a slit connecting the points $s = b^\circ$, $s = -b^\circ$ $(b^\circ = b_1/b_2)$ and passing through the infinitely remote point, and we determine the change in the arguments $-\pi \leqslant \theta_\pm \leqslant \pi$ $(\theta_\pm = \arg(b^\circ \pm s))$. The selected branch possesses the following property

$$f^{1/2}(s) \sim -ib_2 s \operatorname{sgn}(\operatorname{Im} s), \quad s \to \infty \tag{3.6}$$

The factorization (2.6) of the matrix $G(s)$ is constructed by means of (2.7) and (2.8), where the solution of the boundary value problem that vanishes as $s \to \infty$ should be taken as $\beta(s)$

$$\beta^+(t) - \beta^-(t) = f^{-1/2}(t)\varepsilon(t), \quad t \in L \tag{3.7}$$

Since $[\theta_\varepsilon(t)]|_L = 2\pi$ and the point of the contour at which the index of the matrix $\varepsilon(t)$ undergoes a discontinuity is the infinitely remote point, we convert expression (2.9) determining the solution of problem (3.7) into a form (more convenient than (2.9) for analysing the solution at infinity)

$$\beta(s) = \zeta_1(s) + \zeta_2(s) + \zeta_3(s) \tag{3.8}$$
$$\zeta_1(s) = \frac{1}{2\pi i}\int_{L_\gamma^-} \frac{\varepsilon(t)}{f^{1/2}(t)} \frac{dt}{t-s} + \frac{1}{2\pi i}\int_{L_\gamma^+} \frac{\varepsilon(t)-\pi i}{f^{1/2}(t)} \frac{dt}{t-s}$$
$$\zeta_2(s) = \frac{1}{2\pi i}\int_{C_\gamma} \frac{\varepsilon(t)}{f^{1/2}(t)} \frac{dt}{t-s}, \quad \zeta_3(s) = \frac{1}{2}\int_{L_\gamma^+} \frac{dt}{f^{1/2}(t)(t-s)}$$

Taking account of the property 2) of the function $\lambda_n(s)$ as well as of (3.3) and (3.4), we obtain

$$\varepsilon(i\tau) \sim {}^1/_2 \ln[-4(\eta\tau^2)^{-1}], \quad \tau \to 0 \tag{3.9}$$
$$\varepsilon(t) = \varepsilon_0(i\tau) + {}^1/_2\pi i(\operatorname{sgn}\tau + 1), \quad i\tau = t \in L_\gamma^\pm$$
$$\varepsilon_0(i\tau) = {}^1/_2 \ln|\lambda_1(i\tau)/\lambda_2(i\tau)| \sim {}^1/_2 e^{-k_0|\tau|}, \quad \tau \to \pm\infty \quad (k_0 > 0) \tag{3.10}$$

and then by using the substitution $t = i\tau$ we arrive at the expression

$$\zeta_1(s) = -\frac{s}{\pi}\int_{|\gamma|}^{\infty} \frac{\varepsilon_0(i\tau)}{f^{1/2}(i\tau)} \frac{d\tau}{\tau^2+s^2} \tag{3.11}$$

Setting $t = |\gamma|e^{i\theta}$ in the integrand for $\zeta_2(s)$ from (3.8) and using formula 2.2.5.23 from /8/ to evaluate the last integral in (3.8), we obtain

$$\zeta_2(s) = -\frac{|\gamma|}{2\pi}\int_{\pi/2}^{3\pi/2} \frac{\varepsilon(|\gamma|e^{i\theta})}{f^{1/2}(|\gamma|e^{i\theta})} \frac{e^{i\theta}\,d\theta}{|\gamma|e^{i\theta}-s} \tag{3.12}$$
$$\zeta_3(s) = \frac{f^{-1/2}(s)}{2}\ln B(s) - \frac{1}{2}\int_0^{|\gamma|} \frac{f^{-1/2}(i\tau)}{\tau+is}\,d\tau$$
$$B(s) = \frac{b_1[f^{1/2}(s)+b_1]}{isb_2[f^{1/2}(s)-isb_2]}, \quad |\arg is| < \pi, \quad |\arg B(s)| < \pi$$

Let $|s| > 0$ first. We pass to the limit $\gamma \to -0$ in equalities (3.11) and (3.12) and obtain

$$\beta(s) = \frac{f^{-1/2}(s)}{2}\ln B(s) - \frac{s}{\pi}\int_0^{\infty} \frac{\varepsilon_0(i\tau)}{f^{1/2}(i\tau)} \frac{d\tau}{\tau^2+s^2} \tag{3.13}$$

We now determine the behaviour of the canonical matrix $\mathbf{X}(s)$ at infinity. Examining the case $\operatorname{Im} s > 0$ and $\operatorname{Im} s < 0$ separately, taking (3.6) into account, we find the behaviour at infinity of the function $\varepsilon_0(i\tau)$ (3.10) from (3.13)

$$f^{1/2}(s)\beta(s) = -\operatorname{sgn}(\operatorname{Im} s)[\ln(is)^{1/2} + \varkappa] + o(1) \tag{3.14}$$
$$s \to \infty, \quad |\arg is| < \pi$$

$$\varkappa = \frac{1}{2}\ln\frac{2b_2}{b_1} + ib_2q, \quad q = -\frac{1}{\pi}\int_0^{\infty} \frac{\varepsilon_0(i\tau)\,d\tau}{f^{1/2}(i\tau)} \tag{3.15}$$

Furthermore, by virtue of the relationships (3.5) and the equality

$$\int_{-i\infty}^{i\infty} \frac{dt}{t-s} = \pm\pi i, \quad s \in D^\pm$$

we obtain from (2.7)

$$\Lambda^\pm(s) = \pm i\exp\left[-\frac{s}{2\pi}\int_0^{\infty} \frac{\ln|\Delta(i\tau)|}{\tau^2+s^2}\,d\tau\right], \quad s \in D^\pm \tag{3.16}$$

Hence we find $\Lambda^\pm(s) \sim \pm i$, $s \to \infty$, $s \in D^\pm$. Now taking account of (3.14) and (2.7), we have

$$X^{\pm}(s) \sim \pm \frac{ie^{\varkappa}}{2}(is)^{1/2}\begin{Vmatrix} 1 & i \\ -i & 1 \end{Vmatrix}, \quad s\to\infty, \quad s\in D^{\pm}, \quad |\arg is| < \pi$$

Taking into account the equality

$$(is)^{1/2} = e^{\mp i\pi/4}(\mp s)^{1/2}, \quad s\in D^{\pm}, \quad |\arg is| < \pi, \quad |\arg(\mp s)| < \pi/2$$

we finally obtain as $s\to\infty$

$$X^{\pm}(s) \sim e^{\varkappa \mp i\pi/4}\frac{(\mp s)^{1/2}}{\mp 2i}\begin{Vmatrix} 1 & i \\ -i & 1 \end{Vmatrix}, \quad s\in D^{\pm}, \quad |\arg(\mp s)| < \pi/2 \tag{3.17}$$

Defining the stress intensity factors $\sigma_\theta(r,0)$ and $\tau_{r\theta}(r,0)$ by using the relationships (2.10), we have

$$\varphi_n^+(s) \sim K_n(-2s)^{-1/2}, \quad s\to\infty, \quad |\arg(-s)| \leqslant \theta_0 < \pi/2$$
$$\varphi_n^-(s) = O(s^{-1/2}), \quad s\to\infty, \quad |\arg s| \leqslant \theta_1 < \pi/2$$

and, therefore, as $s\to\infty$

$$X^+(s)\varphi^+(s) \sim \frac{i}{2\sqrt{2}}e^{\varkappa - i\pi/4}(K_1 + iK_2)J, \quad s\in D^+\setminus L \tag{3.18}$$
$$X^-(s)\varphi^-(s) = O(1)J, \quad s\in D^-\setminus L, \quad J = \begin{Vmatrix} 1 \\ -i \end{Vmatrix}$$

We substitute (2.6) into (3.1), take account of (3.18) and apply Liouville's theorem. Consequently, for $|s|<\infty$ we have the following formulas for the desired solution of problem (3.1):

$$\varphi^+(s) = ia^{-1}[X^+(s)]^{-1}[CJ - \Omega^+(s)], \quad s\in D^+ \tag{3.19}$$
$$\varphi^-(s) = [X^-(s)]^{-1}[CJ - \Omega^-(s)], \quad s\in D^-$$
$$\Omega(s) = \begin{Vmatrix} \Omega_1(s) \\ \Omega_2(s) \end{Vmatrix}, \quad \Omega(s) = -\frac{a}{2\pi}\int_L \frac{X^+(t)\psi_-^{\circ}(t)}{t-s}dt$$

(C is an arbitrary constant).

Satisfying the additional condition (1.2) (for $k=1$), which taking account of the notation (2.2), can be written in the form

$$\varphi_1^-(0) = \varphi_2^-(0)\operatorname{tg}\alpha \tag{3.20}$$

we find from the second equality in (3.19)

$$C = \frac{A_-\Omega_1^-(0) - A_+\Omega_2^-(0)\operatorname{tg}\alpha}{A_- + iA_+\operatorname{tg}\alpha}, \quad A_{\pm} = c_{\pm}(0) + \frac{s_{\pm}(0)}{(\operatorname{tg}\alpha)^{\pm 1}} \tag{3.21}$$
$$c_{\pm}(0) = \operatorname{ch}[b_1\beta^-(0)] \pm l(0)b_1^{-1}\operatorname{sh}[b_1\beta^-(0)] \tag{3.22}$$
$$s_{\pm}(0) = m_{\mp}(0)b_1^{-1}\operatorname{sh}[b_1\beta^-(0)]$$

We will calculate $\beta^-(0)$ (formula (3.13) is obtained under the assumption that $|s|>0$). Setting $s=0$ in (3.11), we find $\zeta_1(0)=0$. To obtain $\zeta_2(0)$ we take (3.9) into account and represent (3.12) in the form

$$\zeta_2(0) = -\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\left[\frac{\varepsilon(|\gamma|e^{i\theta})}{f^{1/2}(|\gamma|e^{i\theta})} - \frac{1}{2b_1}\ln\frac{-4}{\eta\gamma^2e^{2i\theta}}\right]d\theta - \tag{3.23}$$
$$\frac{1}{4\pi b_1}\int_{\pi/2}^{3\pi/2}\ln\frac{-4}{\eta\gamma^2e^{2i\theta}}d\theta, \quad \arg(-e^{-2i\theta})\in[0,2\pi]$$

Taking into account that $\ln[-\exp(-2i\theta)] = 3\pi i - 2\theta i$, we find from (3.23) as $\gamma\to -0$

$$\zeta_2(0) = -\frac{1}{4b_1}\left(\ln\frac{4}{\eta\gamma^2} + i\pi\right) + o(1)$$

We have from the last equality in (3.8)

$$\zeta_3(0) = \frac{1}{2b_2}\int_{|\gamma|}^{\infty}\frac{d\tau}{\tau\sqrt{\tau^2 + (b_1/b_2)^2}} = \frac{1}{4b_1}\ln\frac{4b_1^2}{b_2^2\gamma^2} + o(1), \quad \gamma\to -0$$

We therefore arrive at the relationship

$$\lim_{\gamma\to -0} \beta^-(0) = (2b_1)^{-1}[\ln(b_1\sqrt{\eta}/b_2) - i\pi/2]$$

and hence, taking account of (3.22) we obtain

$$2c_\pm(0) = c_0 + c_0^{-1} \pm l(0)b_1^{-1}(c_0 - c_0^{-1}) \tag{3.24}$$
$$2s_\pm(0) = m_\mp(0)b_1^{-1}(c_0 - c_0^{-1}), \quad c_0 = (1-i)\sqrt{b_1\eta^{1/2}(2b_2)^{-1}}$$

The quantities $\Omega_j^-(0)$ in (3.21) are found by using the theory of residues. In the special case when $\psi(r) \equiv \psi = \|\psi_1, \psi_2\|, \psi_j = \text{const}$, we have

$$\left\|\begin{matrix}\Omega_1^-(0)\\ \Omega_2^-(0)\end{matrix}\right\| = ai\Lambda^+(-1)\left\|\begin{matrix}c_+(-1)\psi_1 + s_+(-1)\psi_2\\ s_-(-1)\psi_1 + c_-(-1)\psi_2\end{matrix}\right\|$$

Here $\Lambda^+(-1)$ is determined by (3.16) while $c_\pm(-1)$, $s_\pm(-1)$ are the relationships (2.8) and (3.13).

We will obtain computational formulas for the stress intensity factors L_{12}, N_{12}. Comparing the asymptotic equality (3.18) with the relationship resulting from (3.19)

$$X^+(s)\varphi^+(s) \sim a^{-1}iCJ, \quad s\to\infty, \quad s\in D^+\setminus L$$

we arrive at the following formula

$$K_1 + iK_2 = 2\sqrt{2}a^{-1}e^{-\varkappa + i\pi/4}C$$

where $\varkappa$ and C are determined by (3.15) and (3.21), respectively. Therefore, the stress intensity factors introduced in Sect.1 have been found $L_{12} = K_1$, $N_{12} = K_2$.

4. **Problem T_{21}.** The boundary value problem of the theory of elasticity formulated in Sect.1, T_{21} is reduced to the Riemann matrix problem (3.1) where

$$b(s) = {}^1/_2 i\,[d_1^{-1}(s, \pi/2 - \alpha)\cos(\pi - 2\alpha)s - d_2^{-1}(s,\alpha)\cos 2\alpha s]$$
$$c(s) = -{}^1/_2 i\,[d_1^{-1}(s, \pi/2 - \alpha) - d_2^{-1}(s,\alpha)]$$

and $l(s)$, $m_\pm(s)$ and $d_k(s,\alpha)$ are defined in (2.5). The characteristic functions $\lambda_1(s)$, $\lambda_2(s)$ possess properties 1) and 3) from (3.2), as well as the following property analogous to property 2) from (3.2):

$$\lambda_1(\gamma) \sim -i\pi\gamma/4, \quad \lambda_2(\gamma) \sim i\pi(\eta\gamma)^{-1}, \quad \gamma\to -0 \tag{4.1}$$
$$\eta = (2\alpha - \sin 2\alpha)(\pi - 2\alpha + \sin 2\alpha) > 0 \quad (0 < \alpha < \pi/2)$$

Therefore

$$[\arg\lambda_1(t)]|_L = -\pi, \quad [\arg\lambda_2(t)]|_L = \pi; \tag{4.2}$$
$$\varkappa_\Delta = 0, \quad \varkappa_e = -1$$

We determine the branch of the functions $\ln\lambda_n(t)$ $(n = 1, 2)$ exactly as in Sect.3. Then (3.4) and (3.5) remain true. However, because of (4.2), in this case L_γ^- and L_γ^+ change places in (3.3).

The solution of the factorization problem (2.6) for the matrix $G(s)$ is governed by (2.7) and (2.8), where the function $\beta(s)$ in (2.8), which is constructed analogously to that mentioned in Sect.3, differs from (3.13) just by the sign in front of the logarithm and by the replacement of b_2 by $-b_2$.

As $s\to\infty$ we find

$$f^{1/2}(s)\beta(s) = -\operatorname{sgn}(\operatorname{Im} s)[\ln(-is)^{1/2} + \varkappa] + o(1),$$
$$|\arg(-is)| < \pi$$

(formula (3.15) remains true for $\varkappa$).

The behaviour of the canonical matrix $X(s)$ at infinity is determined by the asymptotic equality that differs from (3.17) by the replacement of $\varkappa \mp i\pi/4$ by $\varkappa \pm i\pi/4$.

The solution of the Riemann matrix problem (3.1), corresponding to problem T_{21}, is found from (3.19), where the arbitrary constant in (3.19) is determined from condition (1.2) (for $k = 2$), which can be written in the form

$$\varphi_1^-(0) = -\varphi_2^-(0)\operatorname{ctg}\alpha \tag{4.3}$$

Taking (3.20) and (3.21) into account we find an expression for C from condition (4.3) that differs from (3.21) by the replacement of $\operatorname{tg}\alpha$ by $-\operatorname{ctg}\alpha$. When calculating the value of $\beta^-(0)$ in the expression for $c_\pm(0)$ and $s_\pm(0)$ because of (3.22) we take account of properties (4.1). As in Sect.3, we have

$$\lim_{\gamma\to-0} \beta^-(0) = (2b_1)^{-1}[-\ln(b_1\sqrt{\eta}/b_2) - i\pi/2]$$

Therefore, $c_\pm(0), s_\pm(0)$ are determined by equalities (3.24), where

$$c_0 = (1-i)\sqrt{b_2(2b_1\eta^{1/4})^{-1}}$$

The intensity factors L_{21} and N_{21} are found from the formula

$$L_{21} + iN_{21} = 2\sqrt{2}a^{-1}e^{-\varkappa - i\pi/4}C$$

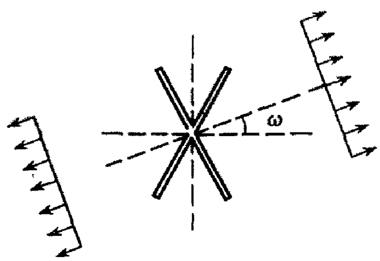

Fig.4

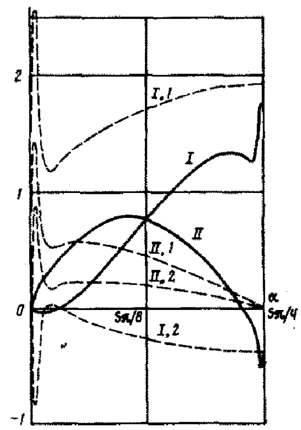

Fig.5

The following plane problem is solved by an analogous scheme. There are two thin absolutely rigid inclusions of identical length intersecting in the middle at an arbitrary angle in the elastic plane. A force P and moment M are applied at an arbitrary angle at the point of intersection of the inclusions. This problem reduces to three separately solvable Riemann matrix problems Z_j with coefficients of the form (2.4), where, if we use the notation $\varkappa_s^{(j)} = \text{ind}\{\lambda_1^{(j)}(t)[\lambda_2^{(j)}(t)]^{-1}\}$, where $\lambda_n^{(j)}(t)$ $(n = 1, 2)$ are eigenfunctions of the matrix coefficient of problem Z_j, then $\varkappa_s^{(1)} = 1$, $\varkappa_s^{(2)} = -1$, $\varkappa_s^{(3)} = 0$.

5. Uniform tension at infinity for a plane with two intersecting slits.

As a numerical example we will consider the problem of the tension in an elastic plane with two intersecting slits by a load of constant intensity p applied at infinity at an angle $\omega = 0$ or $\omega = \pi/4$ to the horizontal axis of symmetry (Fig.4).

In the case when $\omega = 0$

$$L^{(i)} = K_I,\ N^{(i)} = (-1)^{i-1}K_{II}\ (i = 1, 2, 3, 4)$$

where $K_I(K_{II})$ is the intensity factor of the stress $\sigma_\theta(\tau_{r\theta})$ at the apex A_1. The solid lines in Fig.5 represent the dependence of K_I/p and K_{II}/p (curves I and II) on the magnitude of the angle α $(0 < \alpha < \pi/2,\ s = 2)$. We note that for $\alpha = 0$ or $\alpha = \pi/2$ (the case of one crack) $K_{II} = 0$ while $K_I = 0$ for $\alpha = 0$ and $K_I = \sqrt{\pi}p$ for $\alpha = \pi/2$. The factors K_I, K_{II} tend to these limit values as $\alpha \to 0$ and $\alpha \to \pi/2$ (after reaching the minimal value -0.52 for $\alpha = 0.495\pi$ K_{II} starts to increase and tends to zero as $\alpha \to \pi/2$).

In the case when $\omega = \pi/4$

$$L^{(i)} = L^{(i+2)} = K_{I,i},\ N_i = N^{(i+2)} = K_{II,i}\ (i = 1, 2)$$

The dependence of $K_{I,i}/p$, $K_{II,i}/p$ $(i = 1, 2)$ on the quantity α $(0 < \alpha < \pi/4)$ are represented by the dash-dot lines in Fig.5 $(s = 1)$, where

$$K_{I,i}|_{\alpha=\pi/4-\alpha_*} = K_{I,i}|_{\alpha=\pi/4+\alpha_*}, \quad K_{II,i}|_{\alpha=\pi/4-\alpha_*} = -K_{II,i}|_{\alpha=\pi/4+\alpha_*}$$
$$(0 < \alpha_* < \pi/4)$$

In the limit case $\alpha = 0$

$$K_{I,1} = K_{I,2} = \sqrt{\pi}p/2,\ K_{II,1} = K_{II,2} = -\sqrt{\pi}p/2$$

The case of uniform tension with rotation of a plane with two intersecting slits at infinity is also considered.

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A PERTURBATION METHOD FOR MIXED THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY WITH A COMPLEX LINE OF BOUNDARY-CONDITION SEPARATION*

N.M. BORODACHEV

A modification of the perturbation method is proposed, based on the utilization of variational formulas and enabling asymptotic expansions (AE) to be obtained for mixed three-dimensional problems of the theory of elasticity with a complex line of boundary-condition separation. Application of Lighthill's method enables these expansions to be transformed into uniformly suitable ones. The problem for an elastic body with a slit (crack) and the contact problem of the theory of elasticity are considered separately. For the body with a slit the variational formula determines the variation of the displacement of the slit surface caused by variation in the shape of the slit contour. The effectiveness of this formula for constructing AE in problems associated with a perturbation of the shape of the slit contour is shown. Cases of slits of complex shape in an infinite body that differ slightly from a circular slit are examined in detail. A scheme for constructing similar AE is mentioned for spatial contact problems of the theory of elasticity with a complex shape of the contact area.

A review of the application of perturbation methods to mixed problems in the theory of elasticity is contained in /1, 2/. The solutions of mixed spatial problems in the theory of elasticity with a complex line of boundary condition separation, obtained by using other methods, are discussed in /3-8/. The behaviour of the solution of the boundary value problem for a pseudodifferential equation (in particular, crack theory) for variation of the domain was investigated in /9/.

1. We consider a linearly elastic body occupying a simply-connected volume V. Let O be the surface bounding this volume. There is a plane slit of surface S in the body. A kinematic boundary condition is given on the part O_1 of the body surface and a static condition on its other part O_2. The boundary contour of the slit Γ is a plane curve. We use a rectangular system of coordinates x_1, x_2, x_3. The slit is in the plane $x_3 = 0$. We associate the positive orientation S^+ of the surface S with the limit value $x_3 = 0^+$ and the negative orientation S^- with $x_3 = 0^-$. The slit surfaces S^+ and S^- are contained in O_2, i.e., a static boundary condition is given on the slit surface.

Let us magnify the size of the slit by displacing the contour Γ in a nearby location Γ_1. At each point $M \in \Gamma$ we direct the variation $\delta n(M)$ along the outer normal to the curve Γ.

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